

# SEPARATIVE EXCHANGE RINGS IN WHICH 2 IS INVERTIBLE

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## Abstract

An exchange ring  $R$  is separative provided that for all finitely generated projective right  $R$ -modules  $A$  and  $B$ ,  $A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$ . Let  $R$  be a separative exchange ring in which 2 is invertible, and let  $a - a^3 \in R$  be regular. We prove, in this note, that  $a \in R$  is unit-regular if  $R(1 - a^2)R = Rr(a) = \ell(a)$ . An element  $a$  in a ring  $R$  is special clean if there exists an idempotent  $e \in R$  such that  $a - e \in R$  is a unit and  $aR \cap eR = 0$ . Furthermore, we prove that  $a \in R$  is special clean if  $aR/ar(a^2)$ ,  $R/(aR + r(a))$  are projective, and  $R(a - a^3)R = Rar(a^2) = \ell(a^2)aR$ . These also extend the corresponding results in separative regular rings.

**Keywords:** Unit-regular Element; Special Clean Element; Separative Ring; Exchange Ring.

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## 1 Introduction

Let  $R$  be a ring with an identity. An element  $a \in R$  is (unit) regular if there exists some (unit)  $x \in R$  such that  $a = axa$ . A ring  $R$  is (unit) regular if and only if every element in  $R$  is (unit) regular. As is well known, a ring  $R$  is unit-regular if and only if every element in  $R$  is the product of an idempotent and a unit, and that a regular ring is unit-regular if and only if it has stable range one, i.e.,  $A \oplus B \cong A \oplus C \implies B \cong C$  for all finitely generated projective right  $R$ -modules  $A, B$  and  $C$ . Following the terminology used in [1], we say that  $a \in R$  is special clean if there exists an idempotent  $e \in R$  such that  $a - e \in R$  is a unit and  $aR \cap eR = 0$ . Camillo and Khurana Theorem stated that a ring  $R$  is unit-regular if and only if every element in  $R$  is special clean [7, Theorem 1].

A ring  $R$  is an exchange ring if for any  $a \in R$  there exists an idempotent  $e \in aR$  such that  $1 - e \in (1 - a)R$ . The class of exchange rings is very large. It includes all regular rings, all  $\pi$ -regular rings, all strongly  $\pi$ -regular rings, all semiperfect rings, all left or right continuous rings, all clean rings, all unit  $C^*$ -algebras of real rank zero and all right semiartinian rings, etc. A separative ring is one whose finitely generated projective modules satisfy the property  $A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$  [3]. For instances, every weakly

stable exchange ring (including every exchange ring having stable range one) and exchange ring satisfying generalized  $s$ -comparability [6]. Recently, O'Meara proved that the condition  $Rr(a) = \ell(a)R = R(1-a)R$  characterizes elements  $a$  of a regular ring  $R$  that are products of idempotents in precisely the separative regular rings. In fact, one of the most important open problems in regular rings is that if every regular ring is separative [8].

The purpose of this note is to explore when a regular element in separative exchange rings is unit-regular. Let  $R$  be a separative exchange ring in which 2 is invertible, and let  $a - a^3 \in R$  be regular. We prove, in Section 2, that  $a \in R$  is unit-regular if  $R(1-a^2)R = Rr(a) = \ell(a)$ . In Section 3, we further prove that  $a \in R$  is special clean if  $aR/ar(a^2)$ ,  $R/(aR + r(a))$  are projective, and  $R(a - a^3)R = Rar(a^2) = \ell(a^2)aR$ . These also extend [5, Theorem 2.2] and [6, Theorem 15.2.1] from regular rings to exchange ones.

Throughout, all rings are associative with an identity and all modules are right modules. For any right modules  $A$  and  $B$ ,  $A \lesssim^\oplus B$  means that  $A$  is isomorphic to a direct summand of  $B$ . We use  $A \propto B$  to stand for  $A \lesssim^\oplus mB$  for some  $m \in \mathbb{N}$ . We denote the left (right) annihilator of  $a$  in a ring  $R$  by  $\ell(a)$  ( $r(a)$ ).

## 2 Unit-regular Elements

We start by a several lemmas which will be used in the proofs of the main results.

**Lemma 2.1.** [3, Lemma 2.1] *Let  $R$  be an exchange ring. Then the following are equivalent:*

- (1)  $R$  is separative.
- (2) For all finitely generated projective right  $R$ -modules  $A, B$  and  $C$ ,  $C \oplus A \cong C \oplus B$  with  $C \propto A, B \implies A \cong B$ .

**Lemma 2.2.** *Let  $R$  be an exchange ring in which 2 is invertible, and let  $a - a^3 \in R$  be regular. Then*

$$(a - a^3)R \oplus r(a) \cong (a - a^3)R \oplus R/aR.$$

*Proof.* Write  $a - a^3 = (a - a^3)s(a - a^3)$ . Then  $a = axa$ , where  $x = a + (1 - a^2)s(1 - a^2)$ . Set  $b = 1 - a$ . Then  $a = 1 - b$ . Hence,

$$(1 - b)b(2 - b) = (1 - b)b(2 - b)s(1 - b)b(2 - b).$$

As  $2 \in R$  is invertible, we get

$$(1 - b)b(1 - \frac{b}{2}) = (1 - b)b(1 - \frac{b}{2})(2s)(1 - b)b(1 - \frac{b}{2}).$$

Hence,  $b = byb$ , where  $y = \frac{3}{2} - \frac{b}{2} + (1 - b)(1 - \frac{b}{2})(2s)(1 - b)(1 - \frac{b}{2})$ . Set  $c = 1 + a$ . Then

$$c(1 - c)(1 - \frac{c}{2}) = c(1 - c)(1 - \frac{c}{2})(-2s)c(1 - c)(1 - \frac{c}{2}).$$

Thus,  $c = czc$ , where  $z = \frac{3}{2} - \frac{c}{2} + (1 - c)(1 - \frac{c}{2})(-2s)(1 - c)(1 - \frac{c}{2})$ . Clearly,  $r(a) = (1 - xa)R$  and  $r(b) = (1 - yb)R$ . Set  $e = 1 - xa$  and  $f = (1 - yb)$ . Then  $eR + fR = eR + (1 - e)fR$ .

One easily checks that

$$\begin{aligned}
 (1-e)fa(1-e)f &= xa(1-yb)axa(1-yb) \\
 &= xa(1-yb)a(1-yb) \\
 &= xa(1-y(1-a))a(1-y(1-a)) \\
 &= x(1-y(1-a)) \\
 &= (1-e)f.
 \end{aligned}$$

Set  $g = (1-e)fa(1-e)$ . Then  $g = g^2 \in R$  and  $eg = ge = 0$ . It follows that  $eR + fR = (e+g)R$ . Set  $h = e+g$ . Then  $r(a) + r(b) = eR + fR = hR$  with  $h = h^2$ . Further,

$$R = (r(a) + r(b)) \oplus (1-h)R = r(c) \oplus zcR.$$

Since  $R$  is an exchange ring, so is  $\text{end}_R(hR)$ . Thus, we can find some  $C \subseteq r(c)$ ,  $D \subseteq zcR$  such that  $R = (r(a) + r(b)) \oplus C \oplus D$ .

Construct a map  $\varphi : D \rightarrow a(1-a)(1+a)D$  given by  $\varphi(d) = a(1-a)(1+a)d$  for any  $d \in D$ . If  $a(1-a)(1+a)d = 0$ , then  $(1-a)(1+a)d \in r(a)$ ,  $a(1+a)d \in r(b)$  and  $a(1-a)d \in C \oplus D$ . It is easy to verify that

$$\begin{aligned}
 d &= (1-a^2)d + a^2d \\
 &= (1-a)(1+a)d + \frac{1}{2}a(1+a)d - \frac{1}{2}a(1-a)d
 \end{aligned}$$

and so  $d + \frac{1}{2}a(1-a)d \in (C \oplus D) \cap (r(a) \oplus r(b)) = 0$ . This implies that  $d = -\frac{1}{2}a(1-a)d \in r(c) \cap D \subseteq r(c) \cap zcR = 0$ . Therefore  $\varphi : D \rightarrow (a-a^3)D$  is a right  $R$ -isomorphism. Consequently,  $D \cong aD \cong (a-a^3)D = (a-a^3)R$ . Since  $R = r(a) \oplus r(b) \oplus C \oplus D$ ,  $ar(b) = r(b)$  and  $aC = C$ , we see that

$$aR = r(b) \oplus C \oplus D.$$

Hence,

$$\begin{aligned}
 R &= r(b) \oplus C \oplus aD \oplus (1-ax)R \\
 &= r(a) \oplus r(b) \oplus C \oplus D.
 \end{aligned}$$

It follows that  $r(a) \oplus D \cong R/aR \oplus aD$ , and therefore  $(a-a^3)R \oplus r(a) \cong (a-a^3)R \oplus R/aR$ .  $\square$

**Theorem 2.3.** *Let  $R$  be a separative exchange ring in which 2 is invertible, and let  $a-a^3 \in R$  be regular. If  $R(1-a^2)R = Rr(a) = \ell(a)R$ , then  $a \in R$  is unit-regular.*

*Proof.* Suppose  $R(1-a^2)R = Rr(a) = \ell(a)R$ . Construct a map  $\varphi : (1-a^2)R \rightarrow (a-a^3)R$  given by  $\varphi((1-a^2)r) = (a-a^3)r$  for any  $r \in R$ . Clearly,  $\varphi$  is an  $R$ -epimorphism. As  $a-a^3 \in R$  is regular,  $(a-a^3)R$  is projective. Thus,  $\varphi$  splits. So we can find a right  $R$ -module  $D$  such that  $(1-a^2)R \cong (a-a^3)R \oplus D$ . In view of Lemma 2.2,  $(a-a^3)R \oplus r(a) \cong (a-a^3)R \oplus R/aR$ , and then

$$r(a) \oplus (1-a^2)R \cong R/aR \oplus (1-a^2)R.$$

From  $Rr(a) = R(1-a^2)R$ , we get  $(1-a^2)R \subseteq Rr(a)$ . Thus we can find some  $x_1, \dots, x_n \in R$ ;  $y_1, \dots, y_n \in r(a)$  such that  $1-a^2 = \sum_{i=1}^n x_i y_i$ . Write  $a-a^3 = (a-a^3)s(a-a^3)$  for some  $s \in R$ . Then one easily checks that

$$1-a^2 = (1-a^2)(1+a^2sa)(1-a^2).$$

That is,  $1-a^2 \in R$  is regular. So we may assume that each  $x_i \in (1-a^2)R$ . Construct a map  $\varphi : nr(a) \rightarrow (1-a^2)R$  given by  $\varphi(z_1, \dots, z_n) = \sum_{i=1}^n x_i z_i$  for any  $(z_1, \dots, z_n) \in nr(a)$ . Then

$\varphi$  is a right  $R$ -morphism. For any  $(1 - a^2)r \in (1 - a^2)R$ , we have  $(y_1r, \dots, y_nr) \in nr(a)$  such that  $\varphi(y_1r, \dots, y_nr) = (1 - a^2)r$ . This shows that  $\varphi$  is a right  $R$ -epimorphism. As  $(1 - a^2)R$  is projective,  $\varphi$  splits. We conclude that  $(1 - a^2)R \propto r(a)$ .

Write  $aR = gR$  for an idempotent  $g \in R$ . Then  $R/aR \cong (1 - g)R$ . Write  $(1 - a^2)R = eR$  for an idempotent  $e \in R$ . Then  $e \in ReR = \ell(a)R$ . Thus, there are  $x_1, \dots, x_n \in \ell(a)$  and  $r_1, \dots, r_n \in R$  such that  $e = \sum_{i=1}^n x_i r_i$ . Write  $g = ar$ , where  $r \in R$ . Then  $x_i g = x_i ar = 0$ . We infer that  $x_i = x_i(1 - g)$ . Let

$$f = \text{diag}(1 - g, \dots, 1 - g) \in M_n(R), c = e(x_1, \dots, x_n)f, d = f \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} e.$$

Then

$$e = (x_1, \dots, x_n) \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = cd, c = ecf, d = fde.$$

Construct an  $R$ -morphism  $\psi : f(nR) \rightarrow eR, fx \mapsto cx$  for any  $x \in nR$ . Then  $\psi$  is epimorphism. Since  $eR$  is projective, we get  $eR \lesssim^\oplus f(nR)$ . This implies that  $(1 - a^2)R \lesssim^\oplus n((1 - g)R)$ , i.e.,  $(1 - a^2)R \propto R/aR$ . Obviously,  $r(a), R/aR$  are both finitely generated projective right  $R$ -modules, and then we get  $r(a) \cong R/aR$  by Lemma 2.1. Therefore  $a \in R$  is unit-regular, as asserted.  $\square$

**Lemma 2.4.** *Let  $R$  be a separative exchange ring in which 2 is invertible, and let  $a - a^3 \in R$  be regular. If  $(a - a^3)R \propto r(a), R/aR$ , then  $a \in R$  is unit-regular.*

*Proof.* Suppose that  $(a - a^3)R \propto r(a), R/aR$ . By virtue of Lemma 2.2,  $r(a) \oplus (a - a^3)R \cong R/aR \oplus (a - a^3)R$ , where  $r(a), R/aR$  and  $(a - a^3)$  are all finitely generated projective right  $R$ -modules. It follows by Lemma 2.1 that  $r(a) \cong R/aR$ , and therefore  $a \in R$  is unit-regular.  $\square$

**Theorem 2.5.** *Let  $R$  be a separative exchange ring in which 2 is invertible, and let  $a - a^3 \in R$  be regular. If  $R(1 - a^2)R \cap RaR = Rr(a) \cap \ell(a)R \cap RaR$ , then  $a \in R$  is unit-regular.*

*Proof.* Suppose that  $R(1 - a^2)R \cap RaR = Rr(a) \cap \ell(a)R \cap RaR$ . Then we get

$$R(a - a^3)R \subseteq R(1 - a^2)R \cap RaR \subseteq Rr(a),$$

and so  $(a - a^3)R \subseteq Rr(a)$ . As in the proof of Theorem 2.3,  $(1 - a^2)aR \propto r(a)$ . Likewise, we have  $(1 - a^2)aR \propto R/aR$ . This completes the proof by Lemma 2.4.  $\square$

As an immediate consequence, we derive

**Corollary 2.6.** *Let  $R$  be a separative exchange ring in which 2 is invertible. If  $R(1 - a^2)R = Rr(a) \cap \ell(a)R$ , then  $a \in R$  is unit-regular.*

**Remark 2.7.** Let  $R$  be a separative regular ring. O'Meara Theorem proved that  $R(1 - a)R = Rr(a) = \ell(a)R$  implies that  $a \in R$  is a product of idempotents [8, Theorem 4.1]. We ask a question: if 2 is invertible in  $R$ , whether  $R(1 - a^2)R = Rr(a) = \ell(a)R$  (or  $R(1 - a^2)R = Rr(a) \cap \ell(a)R$ ) imply  $a \in R$  is a product of idempotents?

### 3 Special Clean Elements

One easily checks that every special clean element is unit-regular. But the converse is not true. The aim of this section is to explore conditions on a separative exchange ring that 2 is invertible under which a regular element is special clean.

**Lemma 3.1.** [6, Lemma 15.1.1.] *Let  $A$  be a quasi-projective right  $R$ -module. If  $A_1 \subseteq^\oplus A$  and  $A = A_1 + B$ , then there exists some  $A_2 \subseteq B$  such that  $A = A_1 \oplus A_2$ .*

**Lemma 3.2.** *Let  $R$  be a ring, and let  $a \in R$  be regular. If*

- (1)  $aR/ar(a^2)$  is projective;
- (2)  $ar(a^2) \cong R/(r(a) + aR)$ ;

*then  $a \in R$  is special clean.*

*Proof.* Construct  $X, Y, K, Z, e, h$  and  $v$  as in the proof of [6, Lemma 15.1.2]. Then  $a - e \in U(R)$ . Furthermore, we check that

$$aR \bigcap eR \subseteq (K \oplus Z) \bigcap hv(R) \subseteq (K \oplus Z) \bigcap (X \oplus Y) = 0.$$

Therefore  $aR \bigcap eR = 0$ , which completes the proof.  $\square$

**Lemma 3.3.** *Let  $R$  be an exchange ring in which 2 is invertible, and let  $a - a^3 \in R$  be regular. If  $aR/ar(a^2)$  and  $R/(aR + r(a))$  are projective, then*

$$(a - a^3)R \oplus ar(a^2) \cong (a - a^3)R \oplus R/(aR + r(a)).$$

*Proof.* Let  $b = 1 - a$  and  $c = 1 + a$ . As in the proof of Lemma 2.2, there are  $x, y, z \in R$  such that  $a = axa, b = byb$  and  $c = czc$ . Furthermore, we can find some  $C \subseteq r(c), D \subseteq zcR$  such that  $R = (r(a) + r(b)) \oplus C \oplus D$ . Additionally,  $D \cong aD \cong (a - a^3)D = (a - a^3)R$ .

Since  $aR/ar(a^2)$  is projective, the exact sequence

$$0 \rightarrow ar(a^2) \hookrightarrow aR \rightarrow aR/ar(a^2) \rightarrow 0$$

splits, and then there is a right  $R$ -module  $Z$  such that  $aR = ar(a^2) \oplus Z$ . As  $a \in R$  is regular, we see that  $aR$  is projective. Hence,  $ar(a^2)$  is projective, and then so is  $R/(aR + r(a))$ . We infer that

$$0 \rightarrow aR + r(a) \hookrightarrow R \rightarrow R/(aR + r(a)) \rightarrow 0$$

splits. Thus, there exists a right  $R$ -module  $Y$  such that  $R = (aR + r(a)) \oplus Y$ , whence  $aR + r(a)$  is projective. Write  $R = aR \oplus E$ . Then

$$aR + r(a) = (aR + r(a)) \bigcap (aR \oplus E) = aR \oplus (aR + r(a)) \bigcap E,$$

i.e.,  $aR \subseteq^\oplus aR + r(a)$ . By virtue of Lemma 3.1, we have a right  $R$ -module  $X \subseteq r(a)$  such that  $aR + r(a) = aR \oplus X$ . This implies that  $r(a) = r(a) \bigcap (aR \oplus X) = K \oplus X$ , where  $K := r(a) \bigcap aR = ar(a^2)$ . Therefore

$$R = aR \oplus X \oplus Y, aR = K \oplus Z, r(a) = K \oplus X \text{ and } aR + r(a) = aR \oplus X.$$

Hence, we have

$$\begin{aligned}
R &= r(a) \oplus r(b) \oplus C \oplus D \\
&= K \oplus X \oplus r(b) \oplus C \oplus D \\
&= aR \oplus X \oplus Y \\
&= r(b) \oplus aC \oplus aD \oplus X \oplus Y.
\end{aligned}$$

Clearly,  $C = aC$ , and then  $D \oplus K \cong aD \oplus Y$ . This implies that

$$(a - a^3)R \oplus K \cong (a - a^3)R \oplus Y,$$

as asserted.  $\square$

**Theorem 3.4.** *Let  $R$  be a separative exchange ring in which 2 is invertible, and let  $a - a^3 \in R$  be regular. If*

- (1)  $aR/ar(a^2)$  and  $R/(aR + r(a))$  are projective;
- (2)  $Rar(a^2) = \ell(a^2)aR = R(a - a^3)R$ ;

*then  $a \in R$  is special clean.*

*Proof.* Since  $Rar(a^2) = R(a - a^3)R$ , we get  $a(1 - a^2)R \subseteq Rar(a^2)$ . Thus we can find some  $x_1, \dots, x_n \in R; y_1, \dots, y_n \in ar(a^2)$  such that  $a(1 - a^2) = \sum_{i=1}^n x_i y_i$ . As  $a(1 - a^2) \in R$  is regular, we may assume that each  $x_i \in a(1 - a^2)R$ . Construct a map  $\varphi : n(ar(a^2)) \rightarrow a(1 - a^2)R$  given by  $\varphi(z_1, \dots, z_n) = \sum_{i=1}^n x_i z_i$  for any  $(z_1, \dots, z_n) \in n(ar(a^2))$ . Analogously to the proof of Theorem 2.3,  $\varphi$  is a right  $R$ -epimorphism, and therefore  $a(1 - a^2)R \propto ar(a^2)$ .

Write  $aR + r(a) = gR$  for an idempotent  $g \in R$ . Then  $R/(aR + r(a)) \cong (1 - g)R$ . Write  $(a - a^3)R = eR$ . Then  $e \in ReR = \ell(a^2)aR$ . Then we have  $x_1, \dots, x_n \in \ell(a^2)a$  and  $r_1, \dots, r_n \in R$  such that  $e = \sum_{i=1}^n x_i r_i$ . Write  $x_i = s_i a$  for some  $s_i \in \ell(a^2)$ . Then  $s_i a^2 = 0$ . Write  $g = ar + b$ , where  $r \in R, b \in r(a)$ . Then  $x_i g = s_i a(ar + b) = s_i a^2 r = 0$ . We infer that  $x_i = x_i(1 - g)$ . Let

$$f = \text{diag}(1 - g, \dots, 1 - g) \in M_n(R), c = e(x_1, \dots, x_n)f, d = f \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} e.$$

Similarly to the proof of Theorem 2.3, we get an  $R$ -epimorphism  $\psi : f(nR) \rightarrow eR, fx \mapsto cx$  for any  $x \in nR$ . Therefore  $a(1 - a^2)R \lesssim^\oplus n((1 - g)R)$ , i.e.,  $a(1 - a^2)R \propto R/(aR + r(a))$ .

In light of Lemma 3.3,

$$(a - a^3)R \oplus ar(a^2) \cong (a - a^3)R \oplus R/(aR + r(a)).$$

By virtue of Lemma 2.1,  $ar(a^2) \cong R/(r(a) + aR)$ . This completes the proof, in terms of Lemma 3.2.  $\square$

**Corollary 3.5.** *Let  $R$  be a separative regular ring in which 2 is invertible. Then each  $a \in R$  satisfying  $Rar(a^2) = \ell(a^2)aR = R(a - a^3)R$  is special clean.*

*Proof.* Since  $R$  is regular,  $aR, r(a), aR + r(a)$  and  $K = ar(a^2) = aR \cap r(a)$  are direct summands of  $R_R$ . By virtue of Lemma 3.2, we have some  $X \subseteq r(a)$  such that  $aR + r(a) = aR \oplus X$ . Furthermore, there exist  $Y$  and  $Z$  such that  $R = aR \oplus X \oplus Y$  and  $aR = K \oplus Z$ .

Clearly,  $aR/ar(a^2) \cong Z$  and  $R/(aR + r(a)) \cong Y$ . Therefore  $aR/ar(a^2)$  and  $R/(aR + r(a))$  are projective, hence the result, by Theorem 3.4.  $\square$

**Example 3.6.** Let  $V$  be an infinite-dimensional vector space over a field  $\mathbb{Z}_3$ , and let  $R = M_3(\text{end}_{\mathbb{Z}_3}(V))$ . Then  $R$  is weakly stable, and so  $R$  is a separative regular ring by [6,

Theorem 5.2.9]. Choose  $a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R$ . Then  $Rar(a^2) \subseteq R(a - a^3)R$ . Obviously,

$a^2 = a^3$ , and so  $R(a - a^3)R = Ra(1 - a)R \subseteq Rar(a^2)$ . Hence,  $Rar(a^2) = R(a - a^3)R$ . Similarly, we check that  $\ell(a^2)aR = R(a - a^3)R$ . Thus,  $Rar(a^2) = \ell(a^2)aR = R(a - a^3)R$ . Therefore  $a \in R$  is special clean, in terms of Corollary 3.5.

**Theorem 3.7.** Let  $R$  be a separative exchange ring in which 2 is invertible, and let  $a - a^3 \in R$  be regular. If

- (1)  $aR/ar(a^2)$  and  $R/(aR + r(a))$  are projective;
- (2)  $Rar(a^2) = \ell(a^2)aR = R(1 - a^2)R$ ;

then  $a \in R$  is special clean.

*Proof.* By hypothesis,  $Rar(a^2) = \ell(a^2)aR = R(1 - a^2)R$ . Construct a map  $\varphi : (1 - a^2)R \rightarrow (a - a^3)R$  given by  $\varphi((1 - a^2)r) = (a - a^3)r$  for any  $r \in R$ . Then  $\varphi$  is an  $R$ -epimorphism. Analogously to the proof in Theorem 2.3,  $(1 - a^2)R \cong (a - a^3)R \oplus D$  for a right  $R$ -module  $D$ . In light of Lemma 3.3, we get

$$(a - a^3)R \oplus ar(a^2) \cong (a - a^3)R \oplus R/(aR + r(a)).$$

It follows that

$$(1 - a^2)R \oplus ar(a^2) \cong (1 - a^2)R \oplus R/(aR + r(a)).$$

As is the proof of Theorem 3.4, we claim that  $(1 - a^2)R \propto ar(a^2), R/(aR + r(a))$ . Since  $R$  is separative exchange ring, we get  $ar(a^2) \cong R/(aR + r(a))$ . According to Lemma 3.2,  $a \in R$  is special clean.  $\square$

**Corollary 3.8.** Let  $R$  be a separative regular ring in which 2 is invertible. Then each  $a \in R$  satisfying  $Rar(a^2) = \ell(a^2)aR = R(1 - a^2)R$  is special clean.

*Proof.* As in the proof of Corollary 3.5, we prove that  $aR/ar(a^2)$  and  $R/(aR + r(a))$  are projective. Therefore the result follows by Theorem 3.7.  $\square$

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